

Proton Polarizability Contribution: Muonic Hydrogen Lamb Shift and Elastic Scattering

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Abstract

The uncertainty in the contribution to the Lamb shift in muonic hydrogen, ΔE^{subt} arising from proton polarizability effects in the two-photon exchange diagram at large virtual photon momenta is shown large enough to account for the proton radius puzzle. This is because ΔE^{subt} is determined by an integrand that falls very slowly with very large virtual photon momenta. We evaluate the necessary integral using a set of chosen form factors and also a dimensional regularization procedure which makes explicit the need for a low energy constant. The consequences of our two-photon exchange interaction for low-energy elastic lepton-proton scattering are evaluated and could be observable in a planned low energy lepton-proton scattering experiment planned to run at PSI.

I. INTRODUCTION

The proton radius puzzle is one of the most perplexing physics issues of recent times. The extremely precise extraction of the proton radius [1] from the measured energy difference between the $2P_{3/2}^{F=2}$ and $2S_{1/2}^{F=1}$ states of muonic hydrogen disagrees with that extracted from electronic hydrogen. The extracted value of the proton radius is smaller than the CODATA [2] value (based mainly on electronic H) by about 4% or 5.0 standard deviations. This implies [1] that either the Rydberg constant has to be shifted by 4.9 standard deviations or that present QED calculations for hydrogen are insufficient. The Rydberg constant is extremely well measured and the QED calculations seem to be very extensive and highly accurate, so the muonic H finding is a significant puzzle for the entire physics community.

Pohl *et al.* show that the energy difference between the $2P_{3/2}^{F=2}$ and $2S_{1/2}^{F=1}$ states, $\Delta\tilde{E}$ is given by

$$\Delta\tilde{E} = 209.9779(49) - 5.2262r_p^2 + 0.0347r_p^3 \text{ meV}, \quad (1)$$

where r_p is given in units of fm. Using this equation and the experimentally measured value $\Delta\tilde{E} = 206.2949 \text{ meV}$, one can see that the difference between the Pohl and CODATA values of the proton radius would be removed by an increase of the first term on the rhs of Eq. (1) by $0.31 \text{ meV} = 3.1 \times 10^{-10} \text{ MeV}$.

This proton radius puzzle has been attacked from many different directions [3]-[21]. The present communication is intended to investigate the hypothesis that the proton polarizability contributions entering in the two-photon exchange term, see Fig. 1, can account for the 0.31 meV. This idea is worthy of consideration because the computed effect is proportional to the lepton mass to the fourth power, and so is capable of being relevant

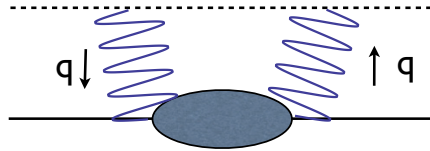


FIG. 1: The box diagram for the $\mathcal{O}(\alpha^5 m^4)$ corrections. The graph in which the photons cross is also included.

for muonic atoms, but irrelevant for electronic atoms.

II. ΔE^{subt} AND ITS EVALUATION

The basic idea is that the two-photon exchange term depends on the forward virtual Compton scattering amplitude $T^{\mu\nu}(\nu, q^2)$ where q^2 is the square of the four momentum, q^μ of the virtual photon and ν is its time component. One uses symmetries to decompose $T^{\mu\nu}(\nu, q^2)$, into a linear combination of two terms, $T_{1,2}(\nu, q^2)$. The imaginary parts of $T_{1,2}(\nu, q^2)$ are related to structure functions $F_{1,2}$ measured in electron- or muon-proton scattering, so that $T_{1,2}$ can be expressed in terms of $F_{1,2}$ through dispersion relations. However, $F_1(\nu, Q^2)$ falls off too slowly for large values of ν for the dispersion relation to converge. Hence, one makes a subtraction at $\nu = 0$, requiring that an additional function of Q^2 (the subtraction function) be introduced. One accounts for the nucleon Born terms, and the remainder of the unknown subtraction function is written as $\bar{T}_1(0, Q^2)$ [17]. This term is handled by making a power series expansion around $Q^2 = 0$, and then using effective field theory to determine the coefficients of the series. The problem with using this expansion is that this contribution to the energy is determined by an integral over all values of Q^2 . We proceed by elaborating the consequences of the behavior of $\bar{T}_1(0, Q^2)$ for large values of Q^2 . This is followed by the development of an alternate effective field theory approach to the muon-proton scattering amplitude. In either case, one can account for the needed Lamb shift, while also providing consequences for the two-photon exchange contribution to the scattering amplitude that can be tested in an upcoming experiment [22].

The contribution to the Lamb shift that is caused by $\bar{T}_1(0, Q^2)$ is denoted as ΔE^{subt} and is given by [17, 18, 23–25]

$$\Delta E^{subt} = \frac{\alpha^2}{m} \phi^2(0) \int_0^\infty \frac{dQ^2}{Q^2} h\left(\frac{Q^2}{4m^2}\right) \bar{T}_1(0, Q^2) \quad (2)$$

where $\phi^2(0) = \frac{\alpha^3 m_r^3}{8\pi}$ for the 2S state with $m, (m_r)$ as the lepton (reduced) mass, and

$$h(t) = (1 - 2t) \left(\left(1 + \frac{1}{t}\right)^{1/2} - 1 \right) + 1. \quad (3)$$

The function $h(t)$ is monotonically falling, approaching $1/\sqrt{t}$ for small values of t , and falling as $3/(4t)$ large values of t . The subtraction function $\bar{T}_1(0, Q^2)$ is not available

from experimental measurements, except at the real photon point $Q^2 = 0$. It comes from the excitation of the proton, and can be described, at small values of Q^2 , in terms of the electric (α_E) and magnetic (β_M) polarizabilities. For small values of Q and $\nu = 0$ one sees [23] $\lim_{\nu^2, Q^2 \rightarrow 0} \bar{T}_1(0, Q^2) = \frac{Q^2}{\alpha} \beta_M$. Using this simple linear Q^2 -dependence in Eq. (2) shows that the integral over $\bar{T}_1(0, Q^2)$ converges at the lower limit, but **diverges logarithmically** at the upper limit. Thus obtaining a non-infinite result depends on including an arbitrary form factor that cuts off the integrand for large values of Q^2 or some other renormalization procedure.

We note that $\lim_{Q^2 \rightarrow \infty} \bar{T}_1(0, Q^2)$ can be obtained from the operator production expansion [26, 27]. Using Eq. (2.18) of Ref. [26], neglecting the term proportional to light quark masses, and accounting for different conventions yields $\bar{T}_1(0, Q^2) \sim 2.1 \text{ fm}^{-1}/Q^2$. This $1/Q^2$ behavior removes the putative logarithmic divergence of $\bar{T}_1(0, Q^2)$, but this function is far from determined.

We follow the previous literature by including a form factor defined as F_{loop} . Then

$$\bar{T}_1(0, Q^2) = \frac{\beta_M}{\alpha} Q^2 F_{\text{loop}}(Q^2). \quad (4)$$

Using Eqs. (2,3,4) one finds the energy shift to be

$$\Delta E^{\text{subt}} = \frac{\alpha^2 \phi^2(0)}{m} \frac{\beta_M}{\alpha} \int_0^\infty dQ^2 \left[(1 - 2Q^2/(4m^2)) \left(\sqrt{1 + \frac{4m^2}{Q^2}} - 1 \right) + 1 \right] F_{\text{loop}}(Q^2). \quad (5)$$

The issue here is the arbitrary nature of the function $F_{\text{loop}}(Q^2)$. Pachucki [24] used the dipole form, $\sim 1/Q^4$, often used to characterize the proton electromagnetic form factors. But the subtraction function should not be computed from the proton form factors, because virtual component scattering includes a term in which the photon is absorbed and emitted from the same quark [28]. Carlson and Vanderhaeghen [17] evaluated a loop diagram using a specific model and found a form factor $\sim 1/Q^2 \log Q^2$, leading to a larger contribution to the subtraction term than previous authors. Birse & McGovern [20] use terms up to fourth-order in chiral perturbation theory to find

$$\bar{T}_1^{BM}(0, Q^2) \simeq \frac{\beta_M}{\alpha} Q^2 \left(1 - \frac{Q^2}{M_\beta^2} + \mathcal{O}(Q^4) \right) \rightarrow \frac{\beta_M}{\alpha} Q^2 \frac{1}{\left(1 + \frac{Q^2}{2M_\beta^2} \right)^2}, \quad (6)$$

with $M_\beta = 460 \pm 50$ MeV. They also use the most recent evaluation of β_M , based on a fit to real Compton scattering [29] that finds

$$\beta_M = (3.1 \pm 0.5) \times 10^{-4} \text{ fm}^3, \quad (7)$$

where only statistical and Baldin Sum Rule errors are included. Their result is a negligible $\Delta E^{subt} = 4.1 \mu \text{ eV}$ [20]. The form Eq. (6) achieves the correct $1/Q^2$ asymptotic behavior of $\bar{T}_1(0, Q^2)$ but the coefficient β_M/α is not the same as obtained from the operator product expansion. The coefficient of Eq. (6) is about twice the asymptotic limit obtained by Collins [26].

Previous authors [17, 20] noted the sensitivity of the integrand of Eq. (5) to large values of Q^2 . Our aim here is to more fully explore the uncertainty in the subtraction term that arises from the logarithmic divergence. We shall use a form of $F_{\text{loop}}(Q^2)$ that is consistent with the constraint on the Q^4 term found Birse & McGovern [20]. This is done by postulating a term that begins at order Q^6 in Eq. (4), such as

$$F_{\text{loop}}(Q^2) = \left(\frac{Q^2}{M_0^2} \right)^n \frac{1}{(1 + aQ^2)^N}, \quad n \geq 2, \quad N \geq n + 3, \quad (8)$$

where M_0, a are parameters to be determined. With Eq. (8) the low Q^2 behavior of $\bar{T}_1(0, Q^2)$ is of order Q^6 or greater and it falls as $1/Q^4$ or greater for large values of Q^2 . So far as we know, there are no constraints on the coefficient of the Q^6 term and the $1/Q^4$ term. However, we shall determine the subtraction term's contribution to the Lamb shift as a general function of n, N . We note that β_M is anomalously small due to a cancellation between pion cloud and intermediate Δ terms [30], so that one can use a value ten times larger than appears in Eq. (7) to set the overall scale of the subtraction term. Thus we replace the term β_M of Eq. (4) by a general form of the same dimensions β : $\beta_M \rightarrow \beta$.

The use of Eq. (8) in Eq. (5) allows one to state the expression for the energy shift in closed form as a general function of n, N . We find

$$\Delta E^{subt} = \frac{\alpha^2 \phi^2(0)}{m} \frac{\beta}{\alpha} \left(\frac{1}{aM_0^2} \right)^n J_{n,N}(m^2 a), \quad (9)$$

$$J_{n,N}(m^2 a) \equiv \frac{1}{a} \int_0^\infty dx \frac{x^n}{(1+x)^N} \left[\left(1 - \frac{x}{2m^2 a} \right) \left(\left(1 + \frac{4m^2 a}{x} \right)^{1/2} - 1 \right) + 1 \right]. \quad (10)$$

The integral over x can be obtained in a closed form in terms of hypergeometric functions. However, a much more understandable expression can be obtained by replacing

the bracketed expression in Eq. (10) by its large argument limit ($3m^2a/x$). This approximation is valid over the entire range of the integrand because of the presence of the factor x^n with $n \geq 2$. Then one obtains

$$J_{n,N}(m^2a) \approx 3m^2 \frac{\Gamma(N-n)\Gamma(n)}{\Gamma(N)} = 3m^2 B(N,n), \quad (11)$$

so that

$$\Delta E^{subt} \approx 3\alpha^2 m \phi^2(0) \frac{\beta}{\alpha} \lambda^n B(N,n), \quad \lambda \equiv \frac{1}{M_0^2 a}. \quad (12)$$

Numerical evaluations show that the approximation is accurate to better than a quarter of a percent. The expression Eq. (12) makes clear the m^4 dependence of the contribution to the Lamb shift.

The numerical value of the term ΔE^{subt} depends on $(n, N), \beta$ and the combination $M_0^2 a \equiv \lambda^{-1}$:

$$\Delta E = 3.91 \text{meV fm}^3 \beta \lambda^n B(N,n). \quad (13)$$

If we take $N = 5, n = 2$ so that $B(5,2) = 1/12$, and $\beta = 10^{-3} \text{ fm}^{-3}$, a value of $\lambda = 30.9$ reproduces $E = 0.31 \text{ meV}$. If we take $M_0 = 0.5 \text{ GeV}$ (as in [20]), then $a^{-1} = 15.4 \text{ GeV}^2$, and that the contribution to the integral comes from the region of very high values of Q^2 . Other values of n, N and λ could be used to get the identical contribution to the Lamb shift.

Chiral perturbation theory could be used to determine the terms of order Q^6 and higher in $\bar{T}_1^{BM}(0, Q^2)$, but this procedure is always limited to a finite number of terms. Indeed one could use values of n greater than 2, and still reproduce the needed contribution to the Lamb shift.

The above discussion shows that the current procedure used to estimate the size of the subtraction term is rather arbitrary. This arises because the chiral EFT is being applied to the virtual-photon nucleon scattering amplitude. Another technique would be to develop an effective field theory to determine the short-distance lepton-nucleon amplitude implied by the subtraction term.

III. EFFECTIVE FIELD THEORY FOR THE μp INTERACTION

The previous considerations show that the value of ΔE^{subt} depends heavily on assumptions the behavior of $\bar{T}_1(0, Q^2)$, $(F_{|rmloop})$ for large values Q^2 . This is true even though the leading $1/Q^2$ term is known. The underlying cause of this uncertainty is the would-be logarithmic divergence in the integral of Eq. (5) for the case $F_{loop} = 1$. This is a symptom that some other technique could be used [31]. Another way to proceed is to use an effective field theory (EFT) for the lepton-proton interaction [32]. In EFT, logarithmic divergences identified through dimensional regularization are renormalized away by including a lepton-proton contact interaction in the Lagrangian.

We may handle the divergence using standard dimensional regularization (DR) techniques by evaluating the scattering amplitude of Fig. 1. The term of interest is obtained by including only $\bar{T}_1(0, Q^2)$ of Eq. (3) with $F_{loop} = 1$. We evaluate the loop integral in $d = 4 - \epsilon$ dimensions and obtain the result:

$$\mathcal{M}_2^{DR}(\text{loop}) = \frac{3}{2} i \alpha^2 m \frac{\beta_M}{\alpha} \left[\frac{2}{\epsilon} + \log \frac{\mu^2}{m^2} + \frac{5}{6} - \gamma_E + \log 4\pi \right] \bar{u}_f u_i \bar{U}_f U_i, \quad (14)$$

where lower case spinors represent leptons of mass m , and upper case proton of mass M , q is momentum transferred to the proton, and γ_E is Euler's constant, $0.577216 \dots$.

The result Eq. (14) corresponds to an infinite contribution to the Lamb shift in the limit that ϵ goes to zero. In EFT one removes the divergent piece by adding a lepton-proton contact interaction to the Lagrangian that removes the divergence, replacing it by an unknown finite part. The finite part is obtained by fitting to a relevant piece of data. Here the only relevant data is the 0.31 meV needed to account for the proton radius puzzle. The low energy term contributes

$$\mathcal{M}_2^{DR}(\text{LET}) = iC(\mu), \quad (15)$$

where $C(\mu)$ is chosen such that the sum of the terms of Eq. (14) and Eq. (15), $\equiv \mathcal{M}_2^{DR}$, is finite and independent of the value of μ . Thus we write the resulting scattering amplitude as

$$\mathcal{M}_2^{DR} = i \alpha^2 m \frac{\beta_M}{\alpha} (\lambda + 5/4) \bar{u}_f u_i \bar{U}_f U_i, \quad (16)$$

where λ is determined by fitting to the Lamb shift. Eq. (16) corresponds to using the \overline{MS} scheme because the term $\log(4\pi) - \gamma_E$ is absorbed into λ .

The corresponding contribution to the Lamb shift is given by

$$\Delta E^{DR} = \alpha^2 m \frac{\beta_M}{\alpha} \phi^2(0) (\lambda + 5/4). \quad (17)$$

Setting ΔE^{DR} to 0.31 meV in the above equation requires that $\lambda = 769$, which seems like a large number. However, as noted above, β_M is extraordinarily small [30]. The natural units of polarizability are $\frac{\beta_M}{\alpha} \sim 4\pi/\Lambda_\chi^3$, [33] where $\Lambda_\chi \equiv 4\pi f_\pi$, (f_π is the pion decay constant). Then Eq. (16) becomes

$$\mathcal{M}_2^{DR} = i 3.95 \alpha^2 m \frac{4\pi}{\Lambda_\chi^3} \bar{u}_f u_i \bar{U}_f U_i. \quad (18)$$

The coefficient 3.95 is of natural size. Thus standard EFT techniques result in an effective lepton-proton interaction of natural size that is proportional to the lepton mass. The form of Eq. (18) is not unique. There are other possible operators that reduce to that form in the low-energy, low-momentum regime of relevance here.

The present results, Eq. (12) and Eq. (17) represent an assumption that there is a lepton-proton interaction of standard-model origin, caused by the high-momentum behavior of the virtual scattering amplitude, that is sufficiently large to account for the proton radius puzzle. Fortunately, our hypothesis can be tested in an upcoming low-energy $\mu^\pm p, e^\pm p$ scattering experiment [22] planned to occur at PSI.

IV. LEPTON PROTON SCATTERING AT LOW ENERGIES

Our aim is to determine the consequences of the particular two-photon exchange term for lepton-proton scattering at low energies. Our previous attempt [21] implied very large corrections to quasi-elastic electron-nucleus scattering that are severely in disagreement with experiment [34]. It is necessary to check that a similar large unwanted contribution does not appear here. Thus we provide a prediction for the PSI experiment. It is well-known that two-photon exchange effects in electron-proton scattering are small at low energies. Our contact interaction is proportional to the lepton mass, so it could provide a measurable effect for muon-proton scattering but be ignorable for electron-proton scattering. We shall investigate the two consequences of using form factors (FF) and effective field theory (DR).

The invariant amplitude is given as $\mathcal{M}_{fi} \equiv \mathcal{M}_{fi}^{(1)} + \mathcal{M}_{fi}^{(2)}$ where the superscripts denote the number of photons exchanged. The first term is given by

$$\mathcal{M}_{fi}^{(1)} = \mp i \frac{e^2}{q^2 + i0} \bar{u}_f \gamma_\mu u_i \bar{U}_f \Gamma^\mu U_i, \quad (19)$$

where $u_{i,f}$ represent leptons of mass m , $U_{i,f}$ represent the proton of mass M , and q is momentum transferred to the proton. The minus sign holds for negatively charged muons, and the plus sign for positively charged muons.

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + i \frac{\sigma^{\mu\nu}}{2M} q_\nu F_2(q^2), \quad (20)$$

with $P_f = P_i + q$. The second-order term that arising from the use of Eq. (4) and Eq. (8), which uses form factors (FF) is given by

$$\mathcal{M}_{fi}^{(2)}(FF) = -\frac{e^4}{(2\pi)^4} \int d^4k \frac{1}{k^2 + i0} \frac{\bar{T}_1(0, -k^2)}{(k - q)^2 + i0} \bar{u}_f L u_i \bar{U}_f U_i, \quad (21)$$

where

$$L = \frac{2mk^2 + 4p_i \cdot k \not{k}}{-4(p_i \cdot k)^2 + k^4 + i0}. \quad (22)$$

The second-order amplitude arising from the use of EFT is given above in Eq. (16).

The cross section depends upon the average over initial and sum over final fermion spins, as denoted by an over line. We first obtain $\overline{|\mathcal{M}_{fi}^{(1)}|^2}$. Standard text-book expressions use the Mott form, obtained by ignoring the lepton mass. This is not a good approximation for the muons of the experiment [22] which have momenta ranging between 100 and 200 MeV/c, and our terms of Eq. (21) and Eq. (16) would vanish in that approximation. We find

$$\overline{|\mathcal{M}_{fi}^{(1)}|^2} = [16M^2(\varepsilon\varepsilon' + q^2/4)(F_1^2 - \frac{q^2}{4M^2}F_2^2) + 4G_M^2(q^4/2 + q^2m^2)] \left(\frac{e^2}{q^2 + i0} \right)^2, \quad (23)$$

where θ is the laboratory scattering angle, and $\varepsilon(\varepsilon')$ is the incident (final) lepton laboratory total energy. The present interference term of interest $\Delta \equiv 2\text{Real} [(\mathcal{M}_{fi}^{(1)})^*(\mathcal{M}_{fi}^{(2)})]$ is obtained using standard trace algebra. We find

$$\begin{aligned} \Delta(FF) = & 8MG_E(q^2) \frac{\mp ie^2}{q^2 + i0} \frac{e^4}{(2\pi)^4} \int d^4k \frac{1}{k^2 + i0} \frac{\bar{T}_1(0, -k^2)}{(k - q)^2 + i0} \frac{1}{(-4(p_i \cdot k)^2 + k^4 + i0)} \\ & \times [2m^2k^2(4\varepsilon M + q^2) + 4p_i \cdot k \not{k} \cdot (P_i + P_f)q^2/2 + 2p_i \cdot k \not{k} \cdot (p_i + p_f)(4\varepsilon M + q^2)]. \end{aligned} \quad (24)$$

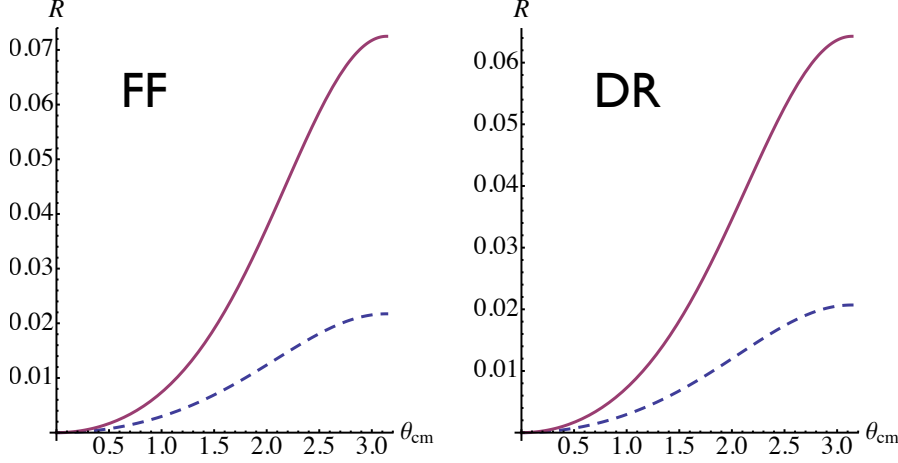


FIG. 2: The ratio R obtained using form factor (FF) regularization or dimensional regularization (DR). The solid curves show the results for muon laboratory momentum of 200 MeV/c and the dashed curves show the results for 100 MeV/c.

We have seen that the integrand is dominated by large values of k , therefore we neglect q in the integrand. This allows considerable simplification so that we find

$$\Delta(F\!F) = \mp 96 G_E(q^2) M^2 \frac{e^2}{q^2 + i0} \epsilon \alpha \beta m^2 \lambda^n B(N, n) \quad (25)$$

where the terms $\lambda, B(N, n)$ appear in Eq. (12). A negligible term proportional to the square of the incident lepton momentum has been dropped. The term Δ adds to the square of the lowest order term for $\mu^- p$ interactions, as expected from an attractive interaction that increases the Lamb shift. The computed value of $\Delta(F\!F)$ does not depend on n, N, λ for those values that reproduce the needed Lamb shift via Eq. (12).

For EFT the contribution to the cross section via interference can be worked out using

Eq. (18) to be

$$\Delta^{DR} = \mp 8[4\varepsilon M + q^2]\alpha(\lambda + \frac{5}{4})m^2\beta_M G_E(q^2)M \frac{e^2}{q^2 + i0}. \quad (26)$$

We are now prepared to display the effects of our two-photon exchange term on $\mu^- - p$ scattering at low energies. The size of the effect is represented by the ratio R , with

$$R \equiv \frac{\Delta}{|\mathcal{M}_{fi}^{(1)}|^2}. \quad (27)$$

The ratio $R > 0$ for $\mu - p$ scattering. The numerator of Eq. (27) is obtained from either Eq. (25) (FF) or Eq. (26) (DR). The ratio R is proportional to the square of the lepton mass, which is negligible for $e^\pm - p$ scattering. We consider two muon momenta 100 and 200 MeV/c. The results are shown in Fig. 2. The angular dependence is dominated by the $Q^2 = -q^2$ term inherent in Eq. (27). The two sets of curves are very similar because the size of the effect is constrained by the required energy shift of 0.31 meV. The size of the effect should be detectable within the expected sub-1 % accuracy of the PSI experiment. We emphasize that our calculation is valid only at low muon laboratory energies.

V. SUMMARY AND DISCUSSION

The findings of this paper can be summarized with a few statements:

- The integrand (see Eq. (2)) that determines the value of ΔE^{subt} values slowly with large values of Q^2 , causing the uncertainty in the evaluation to be large enough to account for the proton radius puzzle.
- The integrand can be evaluated using one of an infinite set of possible form factors or a dimensional regularization procedure.
- Either method can be used to account for the proton radius puzzle and predict an observable effect of a few percent for low energy $\mu - p$ scattering.

The literature [1]-[21] poses several explanations for the proton radius puzzle: The electronic-hydrogen experiments might not be as accurate as previously reported, $\mu - e$ universality might be violated, and that a strong interaction effect entering in a loop diagram is important for muonic hydrogen, but not for electronic hydrogen. It is beyond the

scope of the present paper to argue for the unique correctness of any one of these ideas. The strong-interaction effect discussed here is large enough to be testable experimentally,

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